CURVES AND SURFACES IN EUCLIDEAN SPACE

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1. INTRODUCTION

This article contains a treatment of some of the most elementary theorems in differential geometry in the large. They are the seeds for further developments and the subject should have a promising future. We shall consider the simplest cases, where the geometrical ideas are most clear.

1. THEOREM OF TURNING TANGENTS

Let E be the euclidean plane, which is oriented so that there is a prescribed sense of rotation. We define a smooth curve by expressing its position vector $X = (x_1, x_2)$ as a function of its arc length s. We suppose the function $X(s)$—that is, the functions $x_1(s)$, $x_2(s)$—to be twice continuously differentiable and the vector $X'(s)$ to be nowhere 0. The latter allows the definition of the unit tangent vector $e_1(s)$, which is the unit vector in the direction of $X'(s)$ and, since E is oriented, the unit normal vector $e_2(s)$, so that the rotation from e_1 to e_2 is positive. The vectors $X(s)$, $e_1(s)$, $e_2(s)$ are related by the so-called Frenet formulas

$$(1) \qquad \frac{dX}{ds} = e_1, \qquad \frac{de_1}{ds} = ke_2, \qquad \frac{de_2}{ds} = -ke_1.$$

The function $k(s)$ is called the *curvature*. It is defined together with its sign and changes its sign if the orientation of the curve or of the plane is reversed.

The curve C is called *closed*, if $X(s)$ is periodic of period L, L being the length of C. It is called *simple* if $X(s_1) \neq X(s_2)$, when $0 < s_1 - s_2 < L$. It is said to be *convex* if it lies in one side of every tangent line.

Let C be an oriented closed curve of length L, with the position vector $X(s)$ as a function of the arc length s. Let O be a fixed point in the plane, which we take as the origin of our coordinate system. Denote by Γ the unit circle about O. We define the tangential mapping $T: C \to \Gamma$ as the one which maps a point P of C to the endpoint of the unit vector through O parallel to the tangent vector to C at P. Obviously T is a continuous mapping. It is intuitively clear that when a point goes around C once its image point goes around Γ a number of times. This number will be called the rotation index of C. The theorem of turning tangents asserts that if C is simple, the rotation index is ± 1. We begin by giving a rigorous definition of the rotation index.

We choose a fixed vector through O, say Ox, and denote by $\tau(s)$ the angle which Ox makes with the vector $e_1(s)$. We assume that $0 \leqq \tau(s) < 2\pi$, so that $\tau(s)$ is uniquely determined. This function $\tau(s)$ is, however, not continuous, for in every neighborhood of s_0 at which $\tau(s_0) = 0$ there may be values of $\tau(s)$ differing from 2π by an arbitrarily small quantity. There exists nevertheless a continuous function $\tilde{\tau}(s)$ closely related to $\tau(s)$, as given by the following lemma.

LEMMA: *There exists a continuous function* $\tilde{\tau}(s)$ *such that* $\tilde{\tau}(s) \equiv \tau(s)$, mod 2π.

Proof: To prove the lemma, we observe that the mapping T, being continuous, is uniformly continuous. Therefore, there exists a number $\delta > 0$, such that, for $|s_1 - s_2| < \delta$, $T(s_1)$ and $T(s_2)$ lie in the same open half-plane. From our conditions on $\bar{\tau}(s)$, it follows that, if $\bar{\tau}(s_1)$ is known, $\bar{\tau}(s_2)$ is completely determined. We divide the interval $0 \leqq s \leqq L$ by the points $s_0 \ (= 0) < s_1 < \cdots < s_m \ (= L)$ such that $|s_i - s_{i-1}| < \delta$, $i = 1, \cdots, m$. To define $\bar{\tau}(s)$, we assign to $\bar{\tau}(s_0)$ the value $\tau(s_0)$. Then it is determined in the subinterval $s_0 \leqq s \leqq s_1$, in particular at s_1, which determines it in the second subinterval, etc. The function $\bar{\tau}(s)$ so defined clearly satisfies the conditions of the lemma.

The difference $\bar{\tau}(L) - \bar{\tau}(0)$ is an integral multiple of 2π, say, $= \gamma 2\pi$. We assert that the integer γ is independent of the choice of the function $\bar{\tau}(s)$. In fact, let $\bar{\tau}'(s)$ be a function satisfying the same conditions. Then we have

$$\bar{\tau}'(s) - \bar{\tau}(s) = n(s) \cdot 2\pi,$$

where $n(s)$ is an integer. Since $n(s)$ is continuous in s, it must be a constant. It follows that

$$\bar{\tau}'(L) - \bar{\tau}'(0) = \bar{\tau}(L) - \bar{\tau}(0),$$

which proves the independence of γ from the choice of $\bar{\tau}(s)$. We define γ to be the rotation index of C. The *theorem of turning tangents* follows.

THEOREM: *The rotation index of a simple closed curve is* ± 1.

Proof: To prove this theorem, we consider the mapping Σ which carries an ordered pair of points of C, $X(s_1)$, $X(s_2)$, $0 \leqq s_1 \leqq s_2 \leqq L$, into the endpoint of the unit vector through O parallel to the secant joining $X(s_1)$ to $X(s_2)$. These ordered pairs of points can be represented as a triangle Δ in the (s_1, s_2)-plane defined by $0 \leqq s_1 \leqq s_2 \leqq L$. The mapping Σ of Δ into Γ is continuous. We also observe that its restriction to the side $s_1 = s_2$ is the tangential mapping T.

To a point $p \in \Delta$, let $\tau(p)$ be the angle which Ox makes with $O\Sigma(p)$, such that $0 \leqq \tau(p) < 2\pi$. Again this function need not be continuous. We shall, however, prove that there exists a continuous function $\bar{\tau}(p)$, $p \in \Delta$, such that $\bar{\tau}(p) \equiv \tau(p) \bmod 2\pi$.

In fact, let m be an interior point of Δ. We cover Δ by the radii through m. By the arguments used in the proof of the preceding lemma, we can define a function $\bar{\tau}(p)$, $p \in \Delta$, such that $\bar{\tau}(p) \equiv \tau(p)$, mod 2π, and such that it is continuous along every radius through m. It remains to prove that it is continuous in Δ. For this purpose, let p_0 be a point of Δ. Since Σ is continuous, it follows from the compactness of the segment mp_0 that there exists a number $\eta = \eta(p_0) > 0$, such that, for $q_0 \in mp_0$, and for any point of $q \in \Delta$ for which the distance $d(q, q_0) < \eta$, the points $\Sigma(q)$ and $\Sigma(q_0)$ are never antipodal. The latter condition is equivalent to the relation

$$\bar{\tau}(q) - \bar{\tau}(q_0) \not\equiv 0, \quad \text{mod } \pi. \tag{2}$$

Now let $\epsilon > 0$, $\epsilon < \pi/2$, be given. We choose a neighborhood U of p_0, such that U is contained in the η-neighborhood of p_0, and such that, for $p \in U$, the angle between $O\Sigma(p_0)$ and $O\Sigma(p)$ is less than ϵ. This is possible, because the mapping Σ is continuous. The last condition can be expressed in the form

$$\bar{\tau}(p) - \bar{\tau}(p_0) = \epsilon' + 2k(p)\pi, \quad |\epsilon'| < \epsilon, \tag{3}$$

where $k(p)$ is an integer. Let q_0 be any point on the segment mp_0. Draw the segment q_0q parallel to p_0p, with q on mp. The function $\bar{\tau}(q) - \bar{\tau}(q_0)$ is continuous in q along mp and is zero when q coincides with m. Since $d(q, q_0)$ is less than η, it follows from Equation (2) that $|\bar{\tau}(q) - \bar{\tau}(q_0)| < \pi$. In particular, for $q_0 = p_0$, $|\bar{\tau}(p) - \bar{\tau}(p_0)| < \pi$. Combining this result with Equation (3), we get $k(p) = 0$, which proves that $\bar{\tau}(p)$ is continuous in Δ. Since $\bar{\tau}(p) \equiv \tau(p)$, mod 2π, it is easy to see that $\bar{\tau}(p)$ is differentiable.

Now let $A(0, 0)$, $B(0, L)$, and $D(L, L)$ be the vertices of Δ. The rotation index γ of C is defined by the line integral

$$2\pi\gamma = \int_{AD} d\bar{\tau}.$$

Since $\bar{\tau}(p)$ is defined in Δ, we have

$$\int_{AD} d\bar{\tau} = \int_{AB} d\bar{\tau} + \int_{BD} d\bar{\tau}.$$

To evaluate the line integrals on the right-hand side, we make use of a suitable coordinate system. We can suppose $X(0)$ to be the "lowest" point of C—that is, the point when the vertical coordinate is a minimum, and we choose $X(0)$ to be the origin O. The tangent vector to C at $X(0)$ is horizontal, and we call it Ox. The curve C then lies in the upper half-plane bounded by Ox, and the line integral $\int_{AB} d\bar{\tau}$ is equal to the angle rotated by OP as P traverses once along C. Since OP never points downward, this angle is $\epsilon\pi$, with $\epsilon = \pm 1$. Similarly, the integral $\int_{BD} d\bar{\tau}$ is the angle rotated by PO as P goes once along C. Its value is also equal to $\epsilon\pi$. Hence, the sum of the two integrals is $\epsilon 2\pi$ and the rotation index of C is ± 1, which completes our proof.

We can also define the rotation index by an integral formula. In fact, using the function $\bar{\tau}(s)$ in our lemma, we can express the components of the unit tangent and normal vectors as follows:

$$e_1 = (\cos \bar{\tau}(s), \sin \bar{\tau}(s)), \qquad e_2 = (-\sin \bar{\tau}(s), \cos \bar{\tau}(s)).$$

It follows that

$$d\bar{\tau}(s) = de_1 \cdot e_2 = k\, ds.$$

From this equation, we derive the following formula for the rotation index:

$$2\pi\gamma = \int_C k\, ds. \tag{4}$$

This formula holds for closed curves which are not necessarily simple.

The accompanying figure gives an example of a closed curve with rotation index zero.

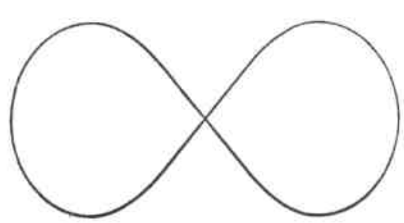

Fig. 1

Many interesting theorems in differential geometry are valid for a more general class of curves, the so-called *sectionally smooth curves*. Such a curve is the union of a finite number of smooth arcs A_0A_1, A_1A_2, $\cdots$, $A_{m-1}A_m$, where the tangents of the two arcs through a common vertex A_i, $i = 1, \cdots, m-1$, may be different. The curve is called *closed*, if $A_0 = A_m$. The simplest example of a closed sectionally smooth curve is a rectilinear polygon.

The notion of rotation index and the theorem of turning tangents can be extended to closed sectionally smooth curves; we summarize, without proof, the result as follows. Let s_i, $i = 1, \cdots, m$, be the arc length measured from A_0 to A_i, so that $s_m = L$ is the length of the curve. The curve supposedly being oriented, the tangential mapping is defined at all points different from A_i. At a vertex A_i there are two unit vectors, tangent respectively to $A_{i-1}A_i$ and A_iA_{i+1}. (We define $A_{m+1} = A_1$.) The corresponding points on Γ we denote by $T(A_i)^-$ and $T(A_i)^+$. Let φ_i be the angle from $T(A_i)^-$ to $T(A_i)^+$, with $0 < \varphi_i < \pi$, briefly the exterior angle from the tangent to $A_{i-1}A_i$ to the tangent to A_iA_{i+1}. For each arc $A_{i-1}A_i$, a continuous function $\bar{\tau}(s)$ can be defined which is one of the determinations of the angle from Ox to the tangent at $X(s)$. The number γ defined by the equation

$$(5) \qquad 2\pi\gamma = \sum_{i=1}^{m} \{\bar{\tau}(s_i) - \bar{\tau}(s_{i-1})\} + \sum_{i=1}^{m} \varphi_i$$

is an integer, which will be called the *rotation index* of the curve. The theorem of turning tangents is again valid.

THEOREM. *If a sectionally smooth curve is simple, the rotation index is equal to* ± 1.

As an application of the theorem of turning tangents, we wish to give the following characterization of a simple closed convex curve.

REMARK: *A simple closed curve is convex, if and only if it can be so oriented that its curvature is greater than, or equal to,* 0.

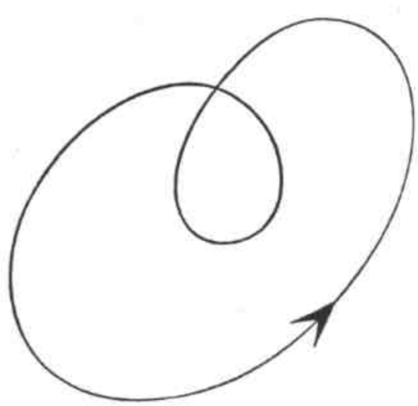

FIG. 2

Let us first remark that the theorem is not true without the assumption that the curve is simple. In fact, the accompanying figure gives a nonconvex curve with $k > 0$.

Proof: To prove the theorem, we let $\bar{\tau}(s)$ be the function constructed, so that we have $k = d\bar{\tau}/ds$. The condition $k \geqq 0$ is equivalent to the assertion that $\bar{\tau}(s)$ is a monotone nondecreasing function. Because C is simple, we can suppose that $\bar{\tau}(s)$, $0 \leqq s \leqq L$, increases from 0 to 2π. It follows that if the tangents at $X(s_1)$ and $X(s_2)$, $0 \leqq s_1 < s_2 < L$, are parallel in the same sense, the arc of C from $X(s_1)$ to $X(s_2)$ is a straight line segment and these tangents must coincide.

Suppose $\bar{\tau}(s)$, $0 \leqq s \leqq L$, is monotone nondecreasing and C is not convex. There is a point $A = X(s_0)$ on C such that there are points of C at both sides of the tangent t to C at A. Choose a positive side of t and consider the oriented perpendicular distance from a point $X(s)$ of C to t. This is a continuous function in s and attains a maximum and a minimum at the points M and N of C, respectively. Clearly M and N are not on t and the tangents to C at M and N are parallel to t. Among these two tangents and t itself, there are two tangents parallel in the same sense, which, according to the preceding remark, is impossible.

Next we let C be convex. To prove that $\bar{\tau}(s)$ is monotone, we suppose $\bar{\tau}(s_1) = \bar{\tau}(s_2)$, $s_1 < s_2$. Then the tangents at $X(s_1)$ and $X(s_2)$ are parallel in the same sense. But there exists a tangent parallel to them in the opposite sense. From the convexity of C it follows that two of them coincide.

We are thus led to the consideration of a line t tangent to C at two distinct points, A and B. We claim that the segment AB must be a part of C. In fact, suppose this is not the case and let D be a point of AB not on C. Draw through D a perpendicular u to t in the half-plane which contains C. Then u intersects C in at least two points. Among these points of intersection, let F be the farthest from t and G the nearest, so that $F \neq G$. Then G is an interior point of the triangle ABF. The tangent to C at G must have points of C in both sides, which contradicts the convexity of C.

It follows that, under the hypothesis of the last paragraph, the segment AB is a part of C and that the tangents at A and B are parallel in the same sense. This proves that the segment joining $X(s_1)$ to $X(s_2)$ belongs to C. The latter implies that $\bar{\tau}(s)$ remains constant in the interval $s_1 \leqq s \leqq s_2$. Hence, the function $\bar{\tau}(s)$ is monotone, and our theorem is proved.

The first half of the theorem can also be stated as follows.

REMARK: *A closed curve with* $k(s) \geqq 0$ *and rotation index equal to* 1 *is convex.*

The theorem of turning tangents was essentially known to Riemann. The above proof was given by H. Hopf, *Compositio Mathematica* 2 (1935), pp. 50–62. For further reading, see:

1. H. Whitney, "On regular closed curves in the plane," *Compositio Mathematica* 4 (1937), pp. 276–84.
2. S. Smale, "Regular curves on a Riemannian manifold," *Transactions of the American Mathematical Society* 87 (1958), pp. 492–511.
3. S. Smale, "A classification of immersions of the two-sphere," *Transactions of the American Mathematical Society* 90 (1959), pp. 281–90.

2. THE FOUR-VERTEX THEOREM

An interesting theorem on closed plane curves is the so-called "four-vertex theorem." By a *vertex* of an oriented closed plane curve we mean a point at which the curvature has a relative extremum. Since the curve forms a compact point set, it has at least two vertices, corresponding respectively to the absolute minimum and maximum of the curvature. Our theorem says that there are at least four.

THEOREM: *A simple closed convex curve has at least four vertices.*

This theorem was first presented by Mukhopadhyaya (1909); the proof we shall give was the work of G. Herglotz. It is also true for nonconvex curves, but the proof is more difficult. The theorem cannot be improved, because an ellipse with unequal axes has exactly four vertices, which are its points of intersection with the axes.

Proof: We suppose that the curve C has only two vertices, M and N, and we shall show that this leads to a contradiction. The line MN does not meet C in any other point, for if it does, the tangent line to C at the middle point must contain the other two points. By the last section, this condition is possible only when the segment MN is a part of C. It would follow that the curvature vanishes at M and N, which is not possible, since they are the points where the curvature takes the absolute maximum and minimum respectively.

We denote by 0 and s_0 the parameters of M and N respectively and take MN to be the x_1-axis. Then we can suppose

$$x_2(s) < 0, \qquad 0 < s < s_0,$$
$$x_2(s) > 0, \qquad s_0 < s < L,$$

where L is the length of C. Let $(x_1(s), x_2(s))$ be the position vector of a point of C with the parameter s. Then the unit tangent and normal vectors have the components

$$e_1 = (x_1', x_2'), \qquad e_2 = (-x_2', x_1'),$$

where primes denote differentiations with respect to s. From the Frenet formulas we get

$$x_1'' = -kx_2', \qquad x_2'' = kx_1'. \tag{6}$$

It follows that

$$\int_0^L kx_2'\,ds = -x_1'\Big|_0^L = 0.$$

The integral in the left-hand side can be written as a sum:

$$\int_0^L kx_2'\,ds = \int_0^{s_0} kx_2'\,ds + \int_{s_0}^L kx_2'\,ds.$$

To each summand we apply the second mean value theorem, which is stated as follows. Let $f(x)$, $g(x)$, $a \leqq x \leqq b$, be two functions in x such that $f(x)$ and $g'(x)$ are continuous and $g(x)$ is monotone. Then there exists ξ, $a < \xi < b$, satisfying the equation,

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^\xi f(x)\,dx + g(b)\int_\xi^b f(x)\,dx.$$

Since $k(s)$ is monotone in each of the intervals $0 \leqq s \leqq s_0$, $s_0 \leqq s \leqq L$, we get

$$\begin{aligned}\int_0^{s_0} kx_2'\,ds &= k(0)\int_0^{\xi_1} x_2'\,ds + k(s_0)\int_{\xi_1}^{s_0} x_2'\,ds \\ &= x_2(\xi_1)(k(0) - k(s_0)), \qquad 0 < \xi_1 < s_0\end{aligned}$$

$$\int_{s_0}^{L} kx_2' \, ds = k(s_0) \int_{s_0}^{\xi_2} x_2' \, ds + k(L) \int_{\xi_2}^{L} x_2' \, ds$$
$$= x_2(\xi_2)(k(s_0) - k(0)), \qquad s_0 < \xi_2 < L.$$

Since the sum of the left-hand members is zero, these equations give

$$(x_2(\xi_1) - x_2(\xi_2))(k(0) - k(s_0)) = 0,$$

which is a contradiction, because

$$x_2(\xi_1) - x_2(\xi_2) < 0, \qquad k(0) - k(s_0) > 0.$$

It follows that there is at least one more vertex on C. Since the relative extrema occur in pairs, there are at least four vertices and the theorem is proved.

At a vertex we have $k' = 0$. Hence, we can also say that on a simple closed convex curve there are at least four points at which $k' = 0$.

The four-vertex theorem is also true for simple closed nonconvex plane curves; see:

1. S. B. Jackson, "Vertices for plane curves," *Bulletin of the American Mathematical Society* 50 (1944), pp. 564–578.

2. L. Vietoris, "Ein einfacher Beweis des Vierscheitelsatzes der ebenen Kurven," *Archiv der Mathematik* 3 (1952), pp. 304–306.

For further reading, see:

1. P. Scherk, "The four-vertex theorem," *Proceedings of the First Canadian Mathematical Congress*. Montreal: 1945, pp. 97–102.

3. ISOPERIMETRIC INEQUALITY FOR PLANE CURVES

The theorem can be stated as follows.

THEOREM: *Among all simple closed curves having a given length the circle bounds the largest area. In other words, if L is the length of a simple closed curve C, and A is the area it bounds, then*

$$L^2 - 4\pi A \geqq 0. \tag{7}$$

Moreover, the equality sign holds only when C is a circle.

Many proofs have been given of this theorem, differing in degree of elegance and in the range of curves under consideration—that is, whether differentiability or convexity is supposed. We shall give two proofs, the work of E. Schmidt (1939) and A. Hurwitz (1902), respectively.

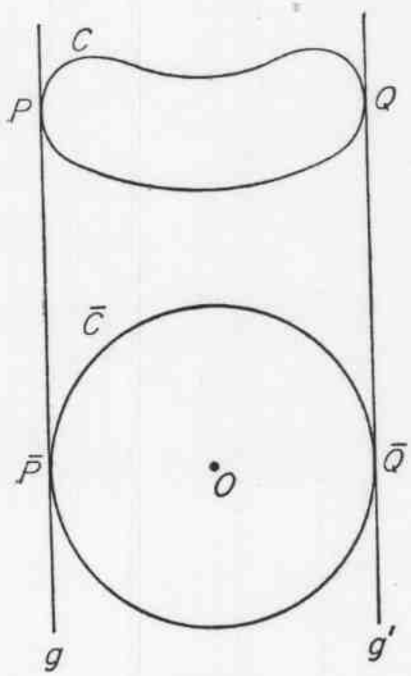

FIG. 3

Schmidt's Proof: We enclose C between two parallel lines, g and g', such that C lies between g and g' and is tangent to them at the points P and Q, respectively. We let $s = 0$, s_0 being the parameters of P and Q, and construct a circle $\overline{C}$ tangent to g and g' at $\overline{P}$ and $\overline{Q}$, respectively. Denote its radius by r and take its center to be the origin of a coordinate system. Let $X(s) = (x_1(s), x_2(s))$ be the position vector of C, so that $(x_1(0), x_2(0)) = (x_1(L), x_2(L))$. As the position vector of $\overline{C}$ we take $(\overline{x}_1(s), x_2(s))$, such that

$$\begin{aligned} \overline{x}_1(s) &= x_1(s), \\ \overline{x}_2(s) &= -\sqrt{r^2 - x_1^2(s)}, \quad 0 \leqq s \leqq s_0 \qquad (8) \\ &= +\sqrt{r^2 - x_1^2(s)}, \quad s_0 \leqq s \leqq L. \end{aligned}$$

Denote by $\overline{A}$ the area bounded by $\overline{C}$. Now the area bounded by a closed curve can be expressed by the line integral

$$A = \int_0^L x_1 x_2' \, ds = -\int_0^L x_2 x_1' \, ds = \tfrac{1}{2} \int_0^L (x_1 x_2' - x_2 x_1') \, ds.$$

Applying this to our two curves C and $\overline{C}$, we get

$$A = \int_0^L x_1 x_2' \, ds$$

$$\overline{A} = \pi r^2 = -\int_0^L \overline{x}_2 \overline{x}_1' \, ds = -\int_0^L \overline{x}_2 x_1' \, ds.$$

Adding these two equations, we have

$$
\begin{aligned}
A + \pi r^2 &= \int_0^L (x_1 x_2' - \bar{x}_2 x_1')\, ds \leqq \int_0^L \sqrt{(x_1 x_2' - \bar{x}_2 x_1')^2}\, ds \\
&\leqq \int_0^L \sqrt{(x_1^2 + \bar{x}_2^2)(x_1'^2 + x_2'^2)}\, ds \qquad (9) \\
&= \int_0^L \sqrt{x_1^2 + \bar{x}_2^2}\, ds = Lr.
\end{aligned}
$$

Since the geometric mean of two positive numbers is less than or equal to their arithmetic mean, it follows that

$$\sqrt{A}\,\sqrt{\pi r^2} \leqq \tfrac{1}{2}(A + \pi r^2) \leqq \tfrac{1}{2}Lr,$$

which gives, after squaring and cancellation of r^2, the inequality in Equation (7).

Suppose now that the equality sign in Equation (7) holds; then A and πr^2 have the same geometric and arithmetic mean, so that $A = \pi r^2$ and $L = 2\pi r$. The direction of the lines g and g' being arbitrary, this means that C has the same "width" in all directions. Moreover, we must have the equality sign everywhere in Equation (9). It follows, in particular, that

$$(x_1 x_2' - \bar{x}_2 x_1')^2 = (x_1^2 + \bar{x}_2^2)(x_1'^2 + x_2'^2),$$

which gives

$$\frac{x_1}{x_2'} = \frac{-\bar{x}_2}{x_1'} = \frac{\sqrt{x_1^2 + \bar{x}_2^2}}{\sqrt{x_1'^2 + x_2'^2}} = \pm r.$$

From the first equality in Equation (9), the factor of proportionality is seen to be r, that is,

$$x_1 = r x_2', \qquad \bar{x}_2 = -r x_1',$$

which remains true when we interchange x_1 and x_2, so that

$$x_2 = r x_1'.$$

Therefore, we have

$$x_1^2 + x_2^2 = r^2,$$

which means that C is a circle.

Hurwitz's proof makes use of the theory of Fourier series. We shall first prove the lemma of Wirtinger.

LEMMA: *Let $f(t)$ be a continuous periodic function of period 2π, possessing a continuous derivative $f'(t)$. If $\int_0^{2\pi} f(t)\,dt = 0$, then*

$$\int_0^{2\pi} f'(t)^2\,dt \geqq \int_0^{2\pi} f(t)^2\,dt. \tag{10}$$

Moreover, the equality sign holds if and only if

$$f(t) = a\cos t + b\sin t. \tag{11}$$

Proof: To prove the lemma, we let the Fourier series expansion of $f(t)$ be

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Since $f'(t)$ is continuous, its Fourier series can be obtained by differentiation term by term, and we have

$$f'(t) \sim \sum_{n=1}^{\infty} (nb_n \cos nt - na_n \sin nt).$$

Since

$$\int_0^{2\pi} f(t)\,dt = \pi a_0,$$

it follows from our hypothesis that $a_0 = 0$. By Parseval's formula, we get

$$\int_0^{2\pi} f(t)^2\,dt = \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

$$\int_0^{2\pi} f'(t)^2\,dt = \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2).$$

Hence,

$$\int_0^{2\pi} f'(t)^2\,dt - \int_0^{2\pi} f(t)^2\,dt = \sum_{n=1}^{\infty} (n^2 - 1)(a_n^2 + b_n^2),$$

which is greater than, or equal to, 0. It is equal to zero, only if $a_n = b_n = 0$ for all $n > 1$. Therefore, $f(t) = a_1 \cos t + b_1 \sin t$, which proves the lemma.

Hurwitz's Proof: In order to prove the inequality in Equation (7), we assume, for simplicity, that $L = 2\pi$, and that

$$\int_0^{2\pi} x_1(s)\,ds = 0.$$

The latter means that the center of gravity lies on the x_1-axis, a condition which can always be achieved by a proper choice of the coordinate system. The length and the area are given by the integrals,

$$2\pi = \int_0^{2\pi} (x_1'^2 + x_2'^2)\, ds, \quad \text{and} \quad A = \int_0^{2\pi} x_1 x_2'\, ds.$$

From these two equations we get

$$2(\pi - A) = \int_0^{2\pi} (x_1'^2 - x_1^2)\, ds + \int_0^{2\pi} (x_1 - x_2')^2\, ds.$$

The first integral is greater than, or equal to, 0 by our lemma and the second integral is clearly greater than, or equal to, 0. Hence, $A \leqq \pi$, which is our isoperimetric inequality.

The equality sign holds only when

$$x_1 = a \cos s + b \sin s, \qquad x_2' = x_1,$$

which gives

$$x_1 = a \cos s + b \sin s, \qquad x_2 = a \sin s - b \cos s + c.$$

Thus, C is a circle.

For further reading, see:

1. E. Schmidt, "Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raum jeder Dimensionenzahl," *Math. Zeit.* 49 (1943), pp. 1–109.

4. TOTAL CURVATURE OF A SPACE CURVE

The *total curvature* of a closed space curve C of length L is defined by the integral

$$\mu = \int_0^L |k(s)|\, ds, \tag{12}$$

where $k(s)$ is the curvature. For a space curve, only $|k(s)|$ is defined.

Suppose C is oriented. Through the origin O of our space we draw vectors of length 1 parallel to the tangent vectors of C. Their end-points describe a closed curve Γ on the unit sphere, to be called the *tangent indicatrix* of C. A point of Γ is singular (that is, with either no tangent or a tangent of higher contact) if it is the image of a point of zero curvature of C. Clearly the total curvature of C is equal to the length of Γ.

Fenchel's theorem concerns the total curvature.

THEOREM: *The total curvature of a closed space curve C is greater than, or equal to, 2π. It is equal to 2π if and only if C is a plane convex curve.*

The following proof of this theorem was found independently by B. Segre (*Bolletino della Unione Matematica Italiana* 13 (1934), 279–283), and by H. Rutishauser and H. Samelson (*Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* 227 (1948), 755–757). See also W. Fenchel, *Bulletin of the American Mathematical Society* 57 (1951), 44–54. The proof depends on the following lemma:

LEMMA: *Let Γ be a closed rectifiable curve on the unit sphere, with length $L < 2\pi$. There exists a point m on the sphere such that the spherical distance $\overline{mx} \leqq L/4$ for all points x of Γ. If Γ is of length 2π but is not the union of two great semicircular arcs, there exists a point m such that $\overline{mx} < \pi/2$ for all x of Γ.*

We use the notation $\overline{ab}$ to denote the spherical distance of two points, a and b. If $\overline{ab} < \pi$, their *midpoint* m is the point defined by the conditions $\overline{am} = \overline{bm} = \frac{1}{2}\overline{ab}$. Let x be a point such that $\overline{mx} \leqq \frac{1}{2}\pi$. Then $2\overline{mx} \leqq \overline{ax} + \overline{bx}$. In fact, let x' be the symmetry of x relative to m. Then,

$$\overline{x'a} = \overline{xb}, \qquad \overline{x'x} = \overline{x'm} + \overline{mx} = 2\overline{mx}.$$

If we use the triangle inequality, it follows that

$$2\overline{mx} = \overline{x'x} \leqq \overline{x'a} + \overline{ax} = \overline{ax} + \overline{bx}, \tag{13}$$

as to be proved.

Lemma Proof: To prove the first part of the lemma, we take two points, a and b, on Γ which divide the curve into two equal arcs. Then $\overline{ab} < \pi$, and we denote the midpoint by m. Let x be a point of Γ such that $2\overline{mx} < \pi$. Such points exist—for example, the point a. Then we have

$$\overline{ax} \leqq \widehat{ax}, \qquad \overline{bx} \leqq \widehat{bx},$$

where $\widehat{ax}$ and $\widehat{bx}$ are respectively the arc lengths along Γ. From Equation (13), it follows that

$$2\overline{mx} \leqq \widehat{ax} + \widehat{bx} = \widehat{ab} = \frac{L}{2}.$$

Hence, the function $f(x) = \overline{mx}$, $x \in \Gamma$, is either $\geqq \pi/2$ or $\leqq L/4 < \pi/2$. Since Γ is connected and $f(x)$ is a continuous function in Γ, the range of the function $f(x)$ is connected in the interval $(0, \pi)$. Therefore, we have $f(x) = \overline{mx} \leqq L/4$.

Consider next the case that Γ is of length 2π. If Γ contains a pair of antipodal points, then, being of length 2π, it must be the union of two great semicircular arcs. Suppose that there is a pair of points, a and b, which bisect Γ such that

$$\overline{ax} + \overline{bx} < \pi$$

for all $x \in \Gamma$. Again, let m denote the midpoint of a and b. If $f(x) = \overline{mx} \leqq \frac{1}{2}\pi$, we have, from Equation (13),

$$2\overline{mx} \leqq \overline{ax} + \overline{bx} < \pi,$$

which means that $f(x)$ omits the value $\pi/2$. Since its range is connected and since $f(a) < \pi/2$, we have $f(x) < \pi/2$ for all $x \in \Gamma$. Thus the lemma is true in this case.

It remains to consider the case that Γ contains no pair of antipodal points, and that for any pair of points a and b which bisect Γ, there is a point $x \in \Gamma$ with

$$\overline{ax} + \overline{bx} = \pi.$$

An elementary geometrical argument, which we leave to the reader, will show that this is impossible. Thus, the lemma is proved.

Theorem Proof: To prove Fenchel's theorem, we take a fixed unit vector A and put

$$g(s) = AX(s),$$

where the right-hand side denotes the scalar product of the vectors A and $X(s)$. The function $g(s)$ is continuous on C and hence must have a maximum and a minimum. Since $g'(s)$ exists, we have, at such an extremum s_0,

$$g'(s_0) = AX'(s_0) = 0.$$

Thus A, as a point on the unit sphere, has a distance $\pi/2$ from at least two points of the tangent indicatrix. Since A is arbitrary, the tangent indicatrix is met by every great circle. It follows from the lemma that its length is greater than, or equal to, 2π.

Suppose next that the tangent indicatrix Γ is of length 2π. By our lemma, it must be the union of two great semicircular arcs. It follows that C itself is the union of two plane arcs. Since C has a tangent everywhere, it must be a plane curve. Suppose C be so oriented that its rotation index

$$\frac{1}{2\pi}\int_0^L k\,ds \geqq 0.$$

Then we have

$$0 \leqq \int_0^L \{|k| - k\}\,ds = 2\pi - \int_0^L k\,ds$$

so that the rotation index is either 0 or 1. To a given vector in the plane there is parallel to it a tangent t of C such that C lies to the left of t. Then t is parallel to the vector in the same sense, and at its point of contact we have $k \geqq 0$, implying that $\int_{k>0} k\,ds \geqq 2\pi$. Since $\int_C |k|\,ds = 2\pi$, there is no point with $k < 0$, and $\int k\,ds = 2\pi$. From the remark at the end of Section 1, we conclude that C is convex.

As a corollary we have the following theorem.

COROLLARY: *If $|k(s)| \leqq 1/R$ for a closed space curve C, C has a length $L \geqq 2\pi R$.*

We have

$$L = \int_0^L ds \geqq \int_0^L R|k|\,ds = R\int_0^L |k|\,ds \geqq 2\pi R.$$

Fenchel's theorem holds also for sectionally smooth curves. As the total curvature of such a curve we define

$$\mu = \int_0^L |k|\,ds + \sum_i a_i \tag{14}$$

where the a_i are the angles at the vertices. In other words, in this case the tangent indicatrix consists of a number of arcs each corresponding to a smooth arc of C; we join successive vertices by the shortest great circular arc on the unit sphere. The length of the curve so obtained is the total curvature of C. It can be proved that for a closed sectionally smooth curve we have also $\mu \geqq 2\pi$.

We wish to give another proof of Fenchel's theorem and a related theorem of Fary-Milnor on the total curvature of a knot.† The basis is Crofton's theorem on the measure of great circles which cut an arc on the unit sphere. Every oriented great circle determines uniquely a "pole," the endpoint of the unit vector normal to the plane of the circle. By the measure of a set of great circles on the unit sphere is meant the area of the domain of their poles. Then Crofton's theorem is stated as follows.

THEOREM: *Let Γ be a smooth arc on the unit sphere Σ_0. The measure of the oriented great circles of Σ_0 which meet Γ, each counted a number of times equal to the number of its common points with Γ, is equal to four times the length of Γ.*

Proof: We suppose Γ is defined by a unit vector $e_1(s)$ expressed as a function of its arc length s. Locally (that is, in a certain neighborhood of s), let $e_2(s)$ and $e_3(s)$ be unit vectors depending smoothly on s, such that the scalar products

$$e_i \cdot e_j = \delta_{ij}, \quad 1 \leqq i, j \leqq 3 \tag{15}$$

and

$$\det(e_1, e_2, e_3) = +1. \tag{16}$$

Then we have

$$\begin{cases} \dfrac{de_1}{ds} = \quad a_2e_2 + a_3e_3, \\[2mm] \dfrac{de_2}{ds} = -a_2e_1 \quad\quad + a_1e_3, \\[2mm] \dfrac{de_3}{ds} = -a_3e_1 - a_1e_2. \end{cases} \tag{17}$$

† I. Fary (*Bulletin de la Société Mathématique de France*, 77 (1949), pp. 128–138), and J. Milnor (*Annals of Mathematics*, 52 (1950), pp. 248–257).

The skew-symmetry of the matrix of the coefficients in the above system of equations follows from differentiation of Equations (15). Since s is the arc length of Γ, we have

$$a_2^2 + a_3^2 = 1, \tag{18}$$

and we put

$$a_2 = \cos\tau(s), \qquad a_3 = \sin\tau(s). \tag{19}$$

If an oriented great circle meets Γ at the point $e_1(s)$, its pole is of the form $Y = \cos\theta\, e_2(s) + \sin\theta \cdot e_3(s)$, and vice versa. Thus (s, θ) serve as local coordinates in the domain of these poles; we wish to find an expression for the element of area of this domain.

For this purpose, we write

$$dY = (-\sin\theta\, e_2 + \cos\theta\, e_3)(d\theta + a_1\, ds) - e_1(a_2\cos\theta + a_3\sin\theta)\, ds.$$

Since $-\sin\theta\, e_2 + \cos\theta\, e_3$ and e_1 are two unit vectors orthogonal to Y, the element of area of Y is

$$|dA| = |a_2\cos\theta + a_3\sin\theta|\, d\theta\, ds = |\cos(\tau - \theta)|\, d\theta\, ds, \tag{20}$$

where the absolute value at the left-hand side means that the area is calculated in the measure-theoretic sense, with no regard to orientation. To the point Y let $Y^\perp$ be the oriented great circle with Y as its pole, and let $n(Y^\perp)$ be the (arithmetic) number of points common to $Y^\perp$ and Γ. Then the measure μ in our theorem is given by

$$\mu = \int n(Y^\perp)|dA| = \int_0^\lambda ds \int_0^{2\pi} |\cos(\tau - \theta)|\, d\theta,$$

where λ is the length of Γ. As θ ranges from 0 to 2π, the variation of $|\cos(\tau - \theta)|$, for a fixed s, is 4. Hence, we get $\mu = 4\lambda$, which proves Crofton's theorem.

By applying the theorem to each subarc and adding, we see that the theorem remains true when Γ is a sectionally smooth curve on the unit sphere. Actually, the theorem is true for any rectifiable arc on the sphere, but the proof is much longer.

For a closed space curve the tangent indicatrix of which fulfills the conditions of Crofton's theorem, Fenchel's theorem is an easy consequence. In fact, the proof of Fenchel's theorem shows us that the tangent indicatrix of a closed space curve meets every great circle in at least two points—that is, $n(Y^\perp) \geqq 2$. It follows that its length is

$$\lambda = \int |k|\, ds = \tfrac{1}{4} \int n(Y^\perp)|dA| \geqq 2\pi,$$

because the total area of the unit sphere is 4π.

Crofton's theorem also leads to the following theorem of Fary and Milnor, which gives a necessary condition on the total curvature of a knot.

THEOREM: *The total curvature of a knot is greater than, or equal to,* 4π.

Since $n(Y^\perp)$ is the number of relative maxima or minima of the "height function," $Y \cdot X(s)$, it is even. Suppose that the total curvature of a closed space curve C is $< 4\pi$. There exists $Y \in \Sigma_0$, such that $n(Y^\perp) = 2$. By a rotation, suppose Y is the point $(0, 0, 1)$. Then the function $x_3(s)$ has only one maximum and one minimum. These points divide C into two arcs, such that x_3 increases on the one and decreases on the other. Every horizontal plane between the two extremal horizontal planes meets C in exactly two points. If we join them by a segment, all these segments will form a surface which is homeomorphic to a circular disk, which proves that C is not knotted.

For further reading, see:

1. S. S. Chern and R. K. Lashof, "On the total curvature of immersed manifolds," I, *American Journal of Mathematics* 79 (1957), pp. 302–18, and II, *Michigan Mathematical Journal* 5 (1958), pp. 5–12.
2. N. H. Kuiper, "Convex immersions of closed surfaces in E^5," *Comm. Math. Helv.* 35 (1961), pp. 85–92.

On integral geometry compare the article of Santalo in this volume.

5. DEFORMATION OF A SPACE CURVE

It is well-known that a one-one correspondence between two curves under which the arc lengths, the curvatures (when not equal to 0), and the torsions are respectively equal, can only be established by a proper motion. It is natural to study the correspondences under which only s and k are equal. We shall call such a correspondence a deformation of the space curve (in German, *Verwindung*). The most notable result in this direction is a theorem of A. Schur, which formulates the geometrical fact that if an arc is "stretched," the distance between its endpoints becomes longer. Using the name curvature to mean here always its absolute value, we state Schur's theorem as follows.

THEOREM: *Let C be a plane arc with the curvature $k(s)$ which forms a convex curve with its chord, AB. Let C^* be an arc of the same length referred to the same parameter s such that its curvature $k^*(s) \leqq k(s)$. If d^* and d denote the lengths of the chords joining their endpoints, then $d \leqq d^*$. Moreover, the equality sign holds when and only when C and C^* are congruent.*

Proof: Let Γ and Γ^* be the tangent indicatrices of C and C^* respectively, P_1 and P_2 two points on Γ, and P_1^* and P_2^* their corresponding points on Γ^*. We denote by $\widehat{P_1P_2}$ and $\widehat{P_1^*P_2^*}$ their arc lengths and by $\overline{P_1P_2}$ and $\overline{P_1^*P_2^*}$ their spherical distances. Then we have

$$\overline{P_1P_2} \leqq \widehat{P_1P_2}, \qquad \overline{P_1^*P_2^*} \leqq \widehat{P_1^*P_2^*}.$$

The inequality on the curvature implies

$$\widehat{P_1^*P_2^*} \leqq \widehat{P_1P_2}. \tag{21}$$

Since C is convex, Γ lies on a great circle, and we have

$$\overline{P_1P_2} = \widehat{P_1P_2},$$

provided that $\overline{P_1P_2} \leqq \pi$. Now let Q be a point on C at which the tangent is parallel to the chord. Denote by P_0 its image point on Γ. Then the condition $\overline{P_0P} \leqq \pi$ is satisfied by any point P on Γ, and if P_0^* denotes the point on Γ^* corresponding to P_0, we have

$$\overline{P_0^*P^*} \leqq \overline{P_0P}, \tag{22}$$

from which it follows that

$$\cos \overline{P_0^*P^*} \geqq \cos \overline{P_0P}, \tag{23}$$

since the cosine function is a monotone decreasing function of its argument when the latter lies between 0 and π.

Because C is convex, d is equal to the projection of C on its chord:

$$d = \int_0^L \cos \overline{P_0P}\, ds. \tag{24}$$

On the other hand, we have

$$d^* \geqq \int_0^L \cos \overline{P_0^*P^*}\, ds, \tag{25}$$

for the integral on the right-hand side is equal to the projection of C^*, and hence of the chord joining its endpoints, on the tangent at the point Q^* corresponding to Q. Combining Equations (23), (24), and (25), we get $d^* \geqq d$.

Suppose that $d = d^*$. Then the inequalities in Equations (22), (23), and (25) become equalities, and the chord joining the endpoints A^* and B^* of C^* must be parallel to the tangent at Q^*. In particular, we have

$$\overline{P_0^*P^*} = \overline{P_0P},$$

which implies that the arcs A^*Q^* and B^*Q^* are plane arcs. On the other hand, we have, by using Equation (21),

$$\overline{P_0^*P^*} \leqq \widehat{P_0^*P^*} \leqq \widehat{P_0P} = \overline{P_0P},$$

or

$$\widehat{P_0^*P^*} = \widehat{P_0P}.$$

Hence, the arcs A^*Q^* and B^*Q^* have the same curvature as AQ and BQ at corresponding points and are therefore respectively congruent.

It remains to prove that the arcs A^*Q^* and B^*Q^* lie in the same plane. Suppose the contrary. They must be tangent at Q^* to the line of intersection of the two distinct planes on which they lie. Since this line is parallel to A^*B^*, the only possibility is that it contains A^* and B^*; however, then the tangent to C at Q must also contain the endpoints A and B, which is a contradiction. Hence, C^* is a plane arc and is congruent to C.

Schur's theorem has many applications. For example, it gives a solution of the following minimum problem: Determine the shortest closed curve with a curvature $k(s) \leqq 1/R$, R being a constant. The answer is, of course, a circle.

Remark: *The shortest closed curve with curvature* $k(s) \leqq 1/R$, R *being a constant, is a circle of radius* R.

By the corollary to Fenchel's theorem, such a curve has length $2\pi R$. Comparing it with a circle of radius R, we conclude from Schur's theorem (with $d^* = d = 0$) that it must itself be a circle.

As a second application of Schur's theorem, we shall derive a theorem of Schwarz. It is concerned with the lengths of arcs joining two given points having a curvature bounded from the above by a fixed constant. The statement of Schwarz's theorem is as follows:

Theorem: *Let* C *be an arc joining two given points* A *and* B, *with curvature* $k(s) \leqq 1/R$, *such that* $R \geqq \frac{1}{2}d$, *where* $d = \overline{AB}$. *Let* S *be a circle of radius* R *through* A *and* B. *Then the length of* C *is either less than, or equal to, the shorter arc* AB *or greater than, or equal to, the longer arc* AB *on* S.

Proof: We remark that the assumption $R \geqq \frac{1}{2}d$ is necessary for the circle S to exist. To prove the theorem, we can assume that the length L of C is less than $2\pi R$; otherwise, there is nothing to prove. We then compare C with an arc of the same length on S having a chord of length d'. The conditions of Schur's theorem are satisfied and we get $d' \leqq d$, d being the distance between A and B. Hence, L is either greater than, or equal to, the longer arc of S with the chord AB, or less than, or equal to, the shorter arc of S with the chord AB.

In particular, we can consider arcs joining A and B with curvature of $1/R$, $R \geqq d/2$. The lengths of such arcs have no upper bound, as shown by the example of a helix. They have d as a lower bound, but can be as close to d as possible. Therefore, we have an example of a minimum problem which has no solution.

Finally, we remark that Schur's theorem can be generalized to sectionally smooth curves. We give here a statement of this generalization without proof.

REMARK: *Let C and C^* be two sectionally smooth curves of the same length, such that C forms a simple convex plane curve with its chord. Referred to the arc length s from one endpoint as parameter, let $k(s)$ be the curvature of C at a regular point and $a(s)$ the angle between the oriented tangents at a vertex; denote corresponding quantities for C^* by the same notations with asterisks. Let d and d^* be the distances between the endpoints of C and C^*, respectively. Then, if*

$$k^*(s) \leqq k(s) \quad \text{and} \quad a^*(s) \leqq a(s),$$

we have $d^* \geqq d$. The equality sign holds if and only if

$$k^*(s) = k(s) \quad \text{and} \quad a^*(s) = a(s).$$

The last set of conditions does not necessarily imply that C and C^* are congruent. In fact, there are simple rectilinear polygons in space which have equal sides and equal angles, but are not congruent.

6. THE GAUSS-BONNET FORMULA

We consider the intrinsic Riemannian geometry on a surface M. To simplify calculations and without loss of generality, we suppose the metric to be given in the isothermal parameters u and v:

$$ds^2 = e^{2\lambda(u,v)}(du^2 + dv^2). \tag{26}$$

The element of area is then

$$dA = e^{2\lambda}\, du\, dv \tag{27}$$

and the area of a domain D is given by the integral

$$A = \iint_D e^{2\lambda}\, du\, dv. \tag{28}$$

Also, the Gaussian curvature of the surface is

$$K = -e^{-2\lambda}(\lambda_{uu} + \lambda_{vv}). \tag{29}$$

It is well-known that the Riemannian metric defines the parallelism of Levi-Civita. To express it analytically, we write

$$u^1 = u \quad \text{and} \quad u^2 = v \tag{30}$$

and

(31) $$ds^2 = \Sigma\, g_{ij}\, du^i\, du^j.$$

In this last formula and throughout this paragraph, our small Latin indices will range from 1 to 2 and a summation sign will mean summation over all repeated indices. From g_{ij} we introduce the g^{ij}, according to the equation

(32) $$\Sigma\, g_{ij} g^{jk} = \delta_i^k$$

and the Christoffel symbols

(33) $$\begin{cases} \Gamma_{ijk} = \dfrac{1}{2}\left(\dfrac{\partial g_{ij}}{\partial u^k} + \dfrac{\partial g_{jk}}{\partial u^i} - \dfrac{\partial g_{ik}}{\partial u^j}\right) \\ \Gamma_{ik}^j = \Sigma\, g^{jh}\Gamma_{ihk} \end{cases}$$

To a vector with the components ξ^i, the Levi-Civita parallelism defines the "covariant differential"

(34) $$D\xi^i = d\xi^i + \Sigma\, \Gamma_{jk}^i\, du^k\, \xi^j.$$

All these equations are well-known in classical Riemannian geometry following the introduction of tensor analysis. The following is a new concept. Suppose the surface M is oriented. Consider the space B of all *unit* tangent vectors of M. This space B is a three-dimensional space, because the set of all unit tangent vectors with the same origin is one-dimensional. (It is called a *fiber space*, meaning that all the unit tangent vectors with origins in a neighborhood form a space which is topologically a product space.) To a unit tangent vector $\xi = (\xi^1, \xi^2)$, let $\eta = (\eta^1, \eta^2)$ be the uniquely determined unit tangent vector, orthogonal to ξ, such that ξ and η form a positive orientation. We introduce the linear differential form

(35) $$\varphi = \sum_{1 \leqq i,j \leqq 2} g_{ij} D\xi^i \eta^j.$$

Then φ is well-defined in B and is usually called the *connection form*.

Because the vector ξ is a unit vector, we can write its components as follows:

(36) $$\xi^1 = e^{-\lambda} \cos\theta \quad \text{and} \quad \xi^2 = e^{-\lambda} \sin\theta.$$

Then

$$\eta^1 = -e^{-\lambda}\sin\theta \quad\text{and}\quad \eta^2 = e^{-\lambda}\cos\theta. \tag{37}$$

Routine calculation gives

$$\begin{aligned}\Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22} = \lambda_u,\\ \Gamma^1_{12} = \Gamma^2_{22} = -\Gamma^2_{11} = \lambda_v,\end{aligned} \tag{38}$$

whence the important relation

$$\varphi = d\theta - \lambda_v\,du + \lambda_u\,dv. \tag{39}$$

Its exterior derivative is therefore

$$d\varphi = -K\,dA. \tag{40}$$

Equation (40) is perhaps the most important formula in two-dimensional local Riemannian geometry.

The connection form φ is a differential form in B. We get from φ a differential form in a subset of M, when there is defined on it a field of unit tangent vectors. For example, let C be a smooth curve on M with the arc length s and let $\xi(s)$ be a smooth unit vector field along C. Then $\varphi = \sigma\,ds$, and σ is called the *variation* of ξ along C. The vectors ξ are said to be parallel along C, if $\sigma = 0$. If ξ is everywhere tangent to C, σ is called the geodesic curvature of C. C is a geodesic of M, if along C the unit tangent vectors are parallel, that is, if its geodesic curvature is 0.

Consider a domain D of M, such that there is a unit vector field defined over D, with an isolated singularity at an interior point $p_0 \in D$. Let γ_ϵ be a circle of geodesic radius ϵ about p_0. Then, from Equation (39), the limit

$$\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\varphi \tag{41}$$

is an integer, to be called the *index* of the vector field at p_0.

Examples of vector fields with isolated singularities are shown in Figure 4. These singularities are, respectively, (a) a source or maximum, (b) a sink or minimum, (c) a center, (d) a simple saddle point, (e) a monkey saddle, and (f) a dipole. The indices are, respectively, 1, 1, 1, -1, -2, and 2.

The Gauss-Bonnet formula is the following theorem.

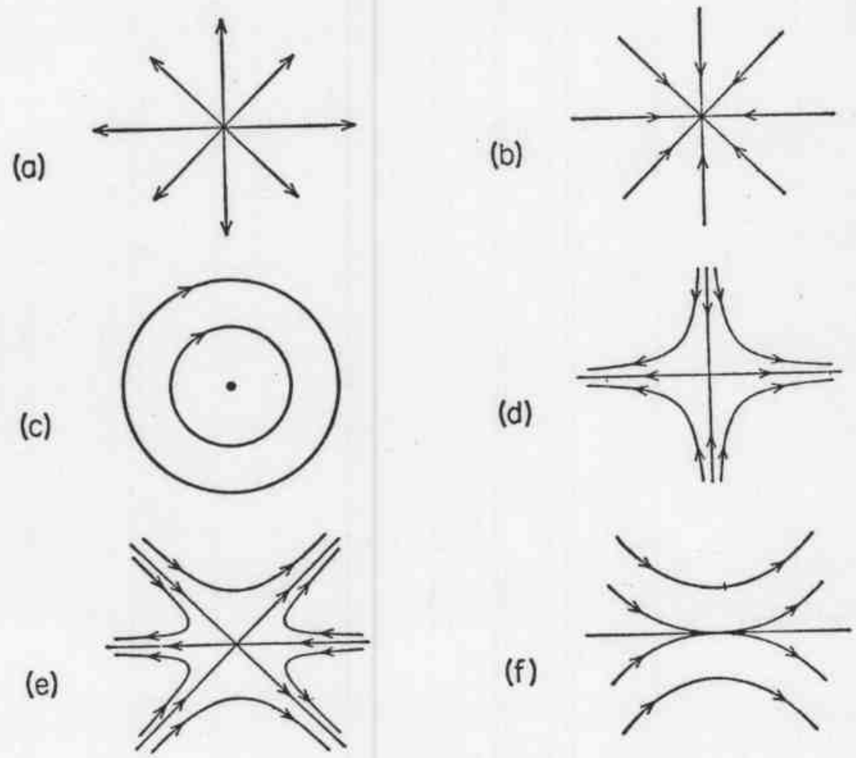

FIG. 4

THEOREM: *Let D be a compact oriented domain in M bounded by a sectionally smooth curve C. Then*

$$\int_C k_g\, ds + \int_D K\, dA + \sum_i (\pi - \alpha_i) = 2\pi\chi, \tag{42}$$

where k_g is the geodesic curvature of C, $\pi - \alpha_i$ are the exterior angles at the vertices of C, and χ is the Euler characteristic of D.

Proof: Consider first the case that D belongs to a coordinate domain (u, v) and is bounded by a simple polygon C of n sides, C_i, $1 \leqq i \leqq n$, with the angles α_i at the vertices. Suppose D is positively oriented. To the points of the arcs C_i we associate the unit tangent vectors to C_i. Thus, to each vertex is associated two vectors at an angle $\pi - \alpha_i$. By the theorem of turning tangents (see Section 1), the total variation of θ as the C_i's are traversed once is $2\pi - \Sigma\, (\pi - \alpha_i)$. It follows that

$$\int_C k_g\, ds = 2\pi - \sum_i (\pi - \alpha_i) + \int_C -\lambda_v\, du + \lambda_u\, dv.$$

By Stokes theorem, the last integral is equal to $-\iint_D K\,dA$. Thus, the formula is proved in this special case.

In the general case, suppose D is subdivided into a union of polygons D_λ, $\lambda = 1, \cdots, f$, such that (1) each D_λ lies in one coordinate neighborhood and (2) two D_λ have either no point, or one vertex, or a whole side, in common. Moreover, let the D_λ be coherently oriented with D, so that every interior side has different senses induced by the two polygons of which it is a side. Let v and e be the numbers of interior vertices and interior sides in this subdivision of D—i.e., vertices and sides which are not on the boundary, C. The above formula can then be applied to each D_λ. Adding all these relations, we have, because the integrals of geodesic curvature along the interior sides cancel,

$$\int_C k_g\,ds + \iint_D K\,dA = 2\pi f - \sum_{i,\lambda} (\pi - \alpha_{\lambda i}) - \sum_i (\pi - \alpha_i)$$

where α_i are the angles at the vertices of D, while the first sum in the right-hand side is extended over all interior vertices of the subdivision. Since each interior side is on exactly two D_λ and since the sum of interior angles about a vertex is 2π, this sum is equal to

$$-2\pi e + 2\pi v.$$

We call the integer

$$\chi(D) = v - e + f \tag{43}$$

the Euler characteristic of D. Substituting, we get Equation (42). Equation (42) has the consequence that the integer χ is independent of the subdivision.

In particular, if C has no vertex, we have

$$\int_C k_g\,ds + \iint_D K\,dA = 2\pi\chi. \tag{44}$$

Moreover, if D is the whole surface M, we get

$$\iint_S K\,dA = 2\pi\chi. \tag{45}$$

It follows that if $K = 0$, the Euler characteristic of M is 0, and M is homeomorphic to a torus. If $K > 0$, then $\chi > 0$, and S is homeomorphic to a sphere.

The Euler characteristic plays an important role in the study of vector fields on a surface.

REMARK: *On a closed orientable surface M, the sum of the indices of a vector field with a finite number of singularities, is equal to the Euler characteristic, $\chi(M)$ of M.*

Proof: Let p_i, $1 \leqq i \leqq n$, be the singularities of the vector field. Let $\gamma_i(\epsilon)$ be a circle of radius ϵ about p_i, and let $\Delta_i(\epsilon)$ be the disk bounded by $\gamma_i(\epsilon)$. Integrating K of A over the domain $M - \bigcup_i \Delta_i(\epsilon)$ and using Equation (40), we get

$$\iint_{M - \bigcup_i \Delta_i(\epsilon)} K\, dA = \sum_i \int_{\gamma_i(\epsilon)} \varphi,$$

where $\gamma_i(\epsilon)$ is oriented so that it is the boundary of $\Delta_i(\epsilon)$. The theorem follows by letting $\epsilon \to 0$.

We wish to give two further applications of the Gauss-Bonnet formula. The first is a theorem of Jacobi. Let $X(s)$ be the coordinate vector of a closed space curve, with the arc length s. Let $T(s)$, $N(s)$, and $B(s)$ be the unit tangent, principal normal, and binormal vectors, respectively. In particular, the curve on the unit sphere with the coordinate vector $N(s)$ is the *principal normal indicatrix*. It has a tangent, wherever

$$k^2 + w^2 \neq 0, \tag{46}$$

where k (when not equal to 0) and w are, respectively, the curvature and torsion of $X(s)$. Jacobi's theorem follows.

THEOREM: *If the principal normal indicatrix of a closed space curve has a tangent everywhere, it divides the unit sphere in two domains of the same area.*

Proof: To prove the theorem, we define τ by the equations

$$k = \sqrt{k^2 + w^2}\cos\tau, \qquad w = \sqrt{k^2 + w^2}\sin\tau. \tag{47}$$

Then we have

$$d(-\cos\tau T + \sin\tau B) = (\sin\tau T + \cos\tau B)\, d\tau - \sqrt{k^2 + w^2}\, N\, ds.$$

Hence, if σ is the arc length of $N(s)$, $d\tau/d\sigma$ is the geodesic curvature of $N(s)$ on the unit sphere. Let D be one of the domains bounded by $N(s)$, and A its area. By the Gauss-Bonnet formula, we have, since $K = 1$,

$$\int_{N(s)} d\tau + \iint dA = 2\pi.$$

It follows that $A = 2\pi$, and the theorem is proved.

Our second application is Hadamard's theorem on convex surfaces.

THEOREM: *If the Gaussian curvature of a closed orientable surface in euclidean space is everywhere positive, the surface is convex (that is, it lies at one side of every tangent plane).*

We discussed a similar theorem for curves in Section 1. For surfaces, it is not necessary to suppose that it has no self-intersection.

Proof: It follows from the Gauss-Bonnet formula that the Euler characteristic $\chi(M)$ of the surface M is positive, so that $\chi(M) = 2$ and

$$\iint_M K\, dA = 4\pi.$$

Suppose M is oriented. We consider the Gauss mapping

$$g: M \to \Sigma_0 \tag{48}$$

(where Σ_0 is the unit sphere about a fixed point 0), which assigns to every point $p \in M$ the end of the unit vector through 0 parallel to the unit normal vector to M at p. The condition $K > 0$ implies that g has everywhere a nonzero functional determinant and is locally one-to-one. It follows that $g(M)$ is an open subset of Σ_0. Since M is compact, $g(M)$ is a compact subset of Σ_0, and hence is also closed. Therefore, g maps onto Σ_0.

Suppose that g is not one-to-one, that is, there exist points p and q of M, $p \neq q$, such that $g(p) = g(q)$. There is then a neighborhood U of q, such that $g(M - U) = \Sigma_0$. Since $\iint_{M-U} K\, dA$ is the area of $g(M - U)$, counted with multiplicities, we will have

$$\iint_{M-U} K\,dA \geqq 4\pi.$$

But

$$\iint_{U} K\,dA > 0,$$

so that

$$\iint_{M} K\,dA = \iint_{U} K\,dA + \iint_{M-U} K\,dA > 4\pi,$$

which is a contradiction, and Hadamard's theorem is proved.

Hadamard's theorem is true under the weaker hypothesis $K \geqq 0$, but the proof is more difficult; see the article by Chern-Lashof mentioned in Section 4.

For further reading, see:

1. S. S. Chern, "On the curvatura integra in a Riemannian manifold," *Annals of Mathematics*, 46 (1945), pp. 674–84.
2. H. Flanders, "Development of an extended exterior differential calculus," *Transactions of the American Mathematical Society*, 75 (1953), pp. 311–26.

See also Section 8 of Flanders's article in this volume.

7. UNIQUENESS THEOREMS OF COHN-VOSSEN AND MINKOWSKI

The "rigidity" theorem of Cohn-Vossen can be stated as follows.

THEOREM: *An isometry between two closed convex surfaces is established either by a motion or by a motion and a reflection.*

In other words, such an isometry is always trivial, and the theorem is obviously not true locally. The following proof is the work of G. Herglotz.

Proof: We shall first discuss some notations on surface theory in euclidean space. Let the surface S be defined by expressing its position vector X as a function of two parameters, u and v. These functions are supposed to be continuously differentiable up to the second order. Suppose that X_u and X_v are everywhere linearly independent, and let ξ be the unit normal vector, so that S is oriented. As usual, let

$$\begin{aligned} \mathrm{I} &= dX\cdot dX = E\,du^2 + 2F\,du\,dv + G\,dv^2 \\ \mathrm{II} &= -dX\cdot d\xi = L\,du^2 + 2M\,du\,dv + N\,dv^2 \end{aligned} \tag{49}$$

be the first and second fundamental forms of the surface. Let H and K denote respectively the mean and Gaussian curvatures.

It is sufficient to prove that under the isometry, the second fundamental forms are equal. Assume the local coordinates are such that corresponding points have the same local coordinates. Then E, F, and G are equal for both surfaces, and the same is true of the Christoffel symbols. Let the second surface be S^*, and denote the quantities pertaining to S^* by the same symbols with asterisks. We introduce

$$\lambda = \frac{L}{D}, \qquad \mu = \frac{M}{D}, \qquad \nu = \frac{N}{D}, \tag{50}$$

where $D = \sqrt{EG - F^2}$. Then the Gaussian curvature is

$$K = \lambda\nu - \mu^2 = \lambda^*\nu^* - \mu^{*2}, \tag{51}$$

and is the same for both surfaces. The mean curvatures are

$$H = \frac{1}{2D}(G\lambda - 2F\mu + E\nu) \quad \text{and} \quad H^* = \frac{1}{2D}(G\lambda^* - 2F\mu^* + E\nu^*). \tag{52}$$

We introduce further

$$J = \lambda\nu^* - 2\mu\mu^* + \nu\lambda^*. \tag{53}$$

The proof depends on the following identity:

$$DJ\xi = \frac{\partial}{\partial u}(\nu^* X_u - \mu^* X_v) - \frac{\partial}{\partial v}(\mu^* X_u - \lambda^* X_v). \tag{54}$$

We first notice that the Codazzi equations can be written in terms of λ^*, μ^*, and ν^* in the form

$$\begin{aligned}\lambda_v^* - \mu_u^* + \Gamma_{22}^2\lambda^* - 2\Gamma_{12}^2\mu^* + \Gamma_{11}^2\nu^* = 0,\\ \mu_v^* - \nu_u^* - \Gamma_{22}^1\lambda^* + 2\Gamma_{12}^1\mu^* - \Gamma_{11}^1\nu^* = 0.\end{aligned} \tag{55}$$

We next write the equations of Gauss:

$$\begin{aligned}X_{uu} - \Gamma_{11}^1 X_u - \Gamma_{11}^2 X_v - D\lambda\xi = 0,\\ X_{uv} - \Gamma_{12}^1 X_u - \Gamma_{12}^2 X_v - D\mu\xi = 0,\\ X_{vv} - \Gamma_{22}^1 X_u - \Gamma_{22}^2 X_v - D\nu\xi = 0.\end{aligned} \tag{56}$$

Multiplying these equations by X_v, $-X_u$, ν^*, $-2\mu^*$, and λ^*, respectively, and adding, we establish Equation (54).

We now write

$$p = Xe_3, \qquad y_1 = XX_u, \qquad y_2 = XX_v, \tag{57}$$

where the right-hand sides are the scalar products of the vectors in question, so that $p(u, v)$ is the oriented distance from the origin to the tangent plane at $X(u, v)$. Equation (54) gives, after taking scalar product with X,

$$\begin{aligned}DJp = -\nu^*E + 2\mu^*F - \lambda^*G\\ + (\nu^*y_1 - \mu^*y_2)_u - (\mu^*y_1 - \lambda^*y_2)_v.\end{aligned} \tag{58}$$

Let C be a closed curve on S. It divides S into two domains, D_1 and D_2, both having C as boundary. Moreover, if D_1 and D_2 are coherently oriented, C appears as a boundary in opposite senses. To each of these domains, say D_1, we apply Green's theorem, and get

$$\begin{aligned}\iint_{D_1} Jp\, dA = \iint_{D_1} (-\nu^*E + 2\mu^*F - \lambda^*G)\, du\, dv\\ + \int_C (+\mu^*y_1 - \lambda^*y_2)\, du + (\nu^*y_1 - \mu^*y_2)\, dv.\end{aligned} \tag{59}$$

Adding this equation to a similar one for D_2, the line integrals cancel, and we have

$$\iint_S Jp\, dA = \iint_S (-\nu^*E + 2\mu^*F - \lambda^*G)\, du\, dv.$$

By Equation (52),

$$\iint_S Jp\, dA = -2 \iint_S H^*\, dA. \tag{60}$$

In particular, this formula is valid when S and S^* are identical, and we have

$$\iint_S 2Kp\,dA = -2\iint_S H\,dA. \tag{61}$$

Subtracting these two equations, we get

$$\iint_S \begin{vmatrix} \lambda^* - \lambda & \mu^* - \mu \\ \mu^* - \mu & \nu^* - \nu \end{vmatrix} p\,dA = 2\iint_S H^*\,dA - 2\iint_S H\,dA. \tag{62}$$

To complete the proof, we need the following elementary lemma.

LEMMA: *Let*

$$ax^2 + 2bxy + cy^2 \quad and \quad a'x^2 + 2b'xy + c'y^2 \tag{63}$$

be two positive definite quadratic forms, with

$$ac - b^2 = a'c' - b'^2. \tag{64}$$

Then

$$\begin{vmatrix} a' - a & b' - b \\ b' - b & c' - c \end{vmatrix} \leqq 0, \tag{65}$$

and the equality sign holds only when the two forms are identical.

As proof, we observe that the statement of the lemma remains unchanged under a linear transformation of the variables. Applying such a linear transformation when necessary, we can assume $b' = b$. Then the left-hand side of Equation (65) becomes

$$(a' - a)(c' - c) = -\frac{c}{a'}(a' - a)^2 \leqq 0,$$

as to be proved. Moreover, the quantity equals 0 only when we also have $a' = a$ and $c' = c$.

We now choose the origin to be inside S, so that $p > 0$. Then the integrand in the left-hand side of Equation (62) is nonpositive, and it follows that

$$\iint_S H^*\,dA \leqq \iint_S H\,dA.$$

Since the relation between S and S^* is symmetrical, we must also have

$$\iint\limits_S H\,dA \leqq \iint\limits_S H^*\,dA.$$

Hence,

$$\iint\limits_S H\,dA = \iint\limits_S H^*\,dA.$$

It follows that the integral at the left-hand side of Equation (62) is 0, and hence, that

$$\lambda^* = \lambda, \qquad \mu^* = \mu, \qquad \nu^* = \nu,$$

completing the proof of Cohn-Vossen's theorem.

By Hadamard's theorem, we see that the Gauss map $g: S \to \Sigma_0$ (see Section 6) is one-to-one for a closed surface with $K > 0$. A point on S can therefore be regarded as a function of its normal vector ξ, and the same is true with any scalar function on S. Minkowski's theorem expresses the unique determination of S when $K(\xi)$ is known.

THEOREM: *Let S be a closed convex surface with Gaussian curvature $K > 0$. The function $K(\xi)$ determines S up to a translation.*

Proof: We shall give a proof of this theorem modeled after the above—that is, by an integral formula [see S. S. Chern, *American Journal of Mathematics* 79 (1957), pp. 949–50]. Let u and v be isothermal parameters on the unit sphere Σ_0, so that we have

$$\xi_u^2 = \xi_v^2 = A > 0 \text{ (say)}, \qquad \xi_u \xi_v = 0. \tag{66}$$

Through the mapping g^{-1} we regard u and v also as parameters on S. Since ξ_u and ξ_v are orthogonal to ξ and are linearly independent, every vector orthogonal to ξ can be expressed as their linear combination. This fact, taken with the relation $X_u \xi_v = X_v \xi_u$, allows us to write

$$\begin{aligned} -X_u &= a\xi_u + b\xi_v, \\ -X_v &= b\xi_u + c\xi_v. \end{aligned} \tag{67}$$

Forming scalar products of these equations with ξ_u and ξ_v, we have

$$Aa = L, \qquad Ab = M, \qquad Ac = N. \tag{68}$$

Moreover, taking the vector product of the two relations in Equation (67), we find

$$X_u \times X_v = (ac - b^2)(\xi_u \times \xi_v).$$

But

$$X_u \times X_v = D\xi, \qquad \xi_u \times \xi_v = A\xi, \tag{69}$$

so that, combining with Equation (68), we have

$$D = A(ac - b^2) = \frac{KD^2}{A},$$

which gives

$$A = KD, \qquad ac - b^2 = \frac{1}{K}. \tag{70}$$

Since $A\,du\,dv$ and $D\,du\,dv$ are, respectively, the volume elements of Σ_0 and S, the first relation in Equation (70) expresses the well-known fact that K is the ratio of these volume elements.

Suppose S^* is another convex surface with the same function, $K(\xi)$. We set up a homeomorphism between S and S^*, so that they have the same normal vector at corrresponding points. Then the parameters u and v can be used for both S and S^*, and corresponding points have the same parameter values. We denote by asterisks the vectors and functions for the surface S^*. Since $K = K^*$, we have from Equation (70), $ac - b^2 = a^*c^* - b^{*2}$ and $D = D^*$.

Let

$$p = X \cdot \xi \quad \text{and} \quad p^* = X^* \cdot \xi \tag{71}$$

be the distances from the origin to the tangent planes of the two surfaces. Our basic relation is the identity

$$\begin{aligned}(X, X^*, X_u)_v &- (X, X^*, X_v)_u \\ &= A\{2(ac - b^2)p^* + (-ac^* - a^*c + 2bb^*)p\} \\ &= A\left\{2(ac - b^2)(p^* - p) + \begin{vmatrix} a - a^* & b - b^* \\ b - b^* & c - c^* \end{vmatrix} p\right\}\end{aligned}$$

which follows immediately from Equations (67), (69), (70), and (71). From it, we find, by Green's theorem, the integral formula

$$(72)\quad \int_{\Sigma_0} \left\{ 2(ac - b^2)(p^* - p) + \begin{vmatrix} a - a^* & b - b^* \\ b - b^* & c - c^* \end{vmatrix} p \right\} A\, du\, dv = 0.$$

By translations if necessary, we can suppose the origin to be inside both surfaces, S and S^*, so that $p > 0$ and $p^* > 0$. Since

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a^* & b^* \\ b^* & c^* \end{pmatrix}$$

are positive definite matrices, it follows from our algebraic lemma that

$$\begin{vmatrix} a - a^* & b - b^* \\ b - b^* & c - c^* \end{vmatrix} \leqq 0.$$

Hence,

$$(73)\qquad \int_{\Sigma_0} (ac - b^2)(p^* - p)\, A\, du\, dv \geqq 0.$$

But the same relation is true when S and S^* are interchanged. Hence, the integral at the left-hand side of Equation (73) must be identically 0. It follows from Equation (72) that

$$\int_{\Sigma_0} \begin{vmatrix} a - a^* & b - b^* \\ b - b^* & c - c^* \end{vmatrix} p\, A\, du\, dv = 0,$$

possible only when $a = a^*$, $b = b^*$, and $c = c^*$. The latter implies that

$$X_u^* = X_u \quad \text{and} \quad X_v^* = X_v,$$

which means that S and S^* differ by a translation.

For further reading, see:

1. S. S. Chern, "Integral formulas for hypersurfaces in euclidean space and their applications to uniqueness theorems," *Journal of Mathematics and Mechanics*, 8 (1959), pp. 947–55.
2. T. Otsuki, "Integral formulas for hypersurfaces in a Riemannian manifold and their applications," *Tôhoku Mathematical Journal*, 17 (1965), pp. 335–48.
3. K. Voss, "Differentialgeometrie geschlossener Flächen im euklidischen Raum," *Jahresberichte deutscher Math. Verein.*, 63 (1960–1961), pp. 117–36.

8. BERNSTEIN'S THEOREM ON MINIMAL SURFACES

A minimal surface is a surface which locally solves the Plateau problem—that is, it is the surface of smallest area bounded by a given closed space curve. Analytically, it is defined by the condition that the mean curvature is identically 0. We suppose the surface to be given by

$$z = f(x, y), \tag{74}$$

where the function $f(x, y)$ is twice continuously differentiable. Then a minimal surface is characterized by the partial differential equation,

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0, \tag{75}$$

where

$$p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial y}, \qquad r = \frac{\partial^2 f}{\partial x^2}, \qquad s = \frac{\partial^2 f}{\partial x \partial y}, \qquad t = \frac{\partial^2 f}{\partial y^2}. \tag{76}$$

Equation (75), called the minimal surface equation, is a nonlinear "elliptic" differential equation.

Bernstein's theorem is the following "uniqueness theorem."

THEOREM: *If a minimal surface is defined by Equation* (74) *for all values of x and y, it is a plane. In other words, the only solution of Equation* (75) *valid in the whole (x, y)-plane is a linear function.*

Proof: We shall derive this theorem as a corollary of the following theorem of Jörgens [*Math Annalen* 127 (1954), pp. 130–34].

THEOREM: *Suppose the function $z = f(x, y)$ is a solution of the equation*

$$rt - s^2 = 1, \quad r > 0, \tag{77}$$

for all values of x and y. Then $f(x, y)$ is a quadratic polynomial in x and y.

For fixed (x_0, y_0) and (x_1, y_1), consider the function

$$h(t) = f(x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0)).$$

We have

$$h'(t) = (x_1 - x_0)p + (y_1 - y_0)q,$$

$$h''(t) = (x_1 - x_0)^2 r + 2(x_1 - x_0)(y_1 - y_0)s + (y_1 - y_0)^2 s \geqq 0,$$

where the arguments in the functions p, q, r, s, t are $x_0 + t(x_1 - x_0)$ and $y_0 + t(y_1 - y_0)$. From the last inequality, it follows that

$$h'(1) \geqq h'(0)$$

or

$$(x_1 - x_0)(p_1 - p_0) + (y_1 - y_0)(q_1 - q_0) \geqq 0. \tag{78}$$

where

$$p_i = p(x_i, y_i) \quad \text{and} \quad q_i = q(x_i, y_i), \quad i = 0, 1. \tag{79}$$

Consider the transformation of Lewy:

$$\xi = \xi(x, y) = x + p(x, y), \qquad \eta = \eta(x, y) = y + q(x, y). \tag{80}$$

Setting

$$\xi_i = \xi(x_i, y_i), \qquad \eta_i = \eta(x_i, y_i), \qquad i = 0, 1, \tag{81}$$

we have, by Equation (78),

$$(\xi_1 - \xi_0)^2 + (\eta_1 - \eta_0)^2 \geqq (x_1 - x_0)^2 + (y_1 - y_0)^2. \tag{82}$$

Hence, the mapping

$$(x, y) \to (\xi, \eta) \tag{83}$$

is distance-increasing.

Moreover, we have

$$\begin{aligned} \xi_x &= 1 + r, & \xi_y &= s \\ \eta_x &= s, & \eta_y &= 1 + t, \end{aligned} \tag{84}$$

so that

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = 2 + r + t \geqq 2, \tag{85}$$

and the mapping in Equation (83) is locally one-to-one. It follows easily that Equation (83) is a diffeomorphism of the (x, y)-plane onto the (ξ, η)-plane.

We can therefore regard the function $f(x, y)$, which is a solution of Equation (77), as a function in ξ and η. Let

$$F(\xi, \eta) = x - iy - (p - iq), \tag{86}$$

$$\zeta = \xi + i\eta. \tag{87}$$

It can be verified by a computation that $F(\xi, \eta)$ satisfies the Cauchy-Riemann equations, so that $F(\zeta) = F(\xi, \eta)$ is a holomorphic function in ζ. Moreover, we have

$$F'(\zeta) = \frac{t - r + 2is}{2 + r + t}. \tag{88}$$

From the last relation, we get

$$1 - |F'(\zeta)|^2 = \frac{4}{2 + r + t} > 0.$$

Thus $F'(\zeta)$ is bounded in the whole ζ-plane. By Liouville's theorem, we have

$$F'(\zeta) = \text{const.}$$

On the other hand, by Equation (88) we have

$$r = \frac{|1 - F'|^2}{1 - |F'|^2}, \qquad s = \frac{i(\overline{F'} - F')}{1 - |F'|^2}, \qquad t = \frac{|1 + F'|^2}{1 - |F'|^2}. \tag{89}$$

It follows that r, s, and t are all constants, and Jörgens's theorem is proved.

Bernstein's theorem is an easy consequence of Jörgens's theorem. In fact, let

$$W = (1 + p^2 + q^2)^{1/2}. \tag{90}$$

Then the minimal surface equation is equivalent to each of the following equations:

$$\begin{aligned} \frac{\partial}{\partial x}\frac{-pq}{W} + \frac{\partial}{\partial y}\frac{1 + p^2}{W} &= 0, \\ \frac{\partial}{\partial x}\frac{1 + q^2}{W} + \frac{\partial}{\partial y}\frac{-pq}{W} &= 0. \end{aligned} \tag{91}$$

It follows that there exists a C^2-function, $\varphi(x, y)$, such that

$$\varphi_{xx} = \frac{1}{W}(1 + p^2), \qquad \varphi_{xy} = \frac{1}{W}pq, \qquad \varphi_{yy} = \frac{1}{W}(1 + q^2). \tag{92}$$

These partial derivatives satisfy the equation

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = 1, \quad \varphi_{xx} > 0.$$

By Jörgens's theorem, φ_{xx}, φ_{xy}, and φ_{yy} are constants. Hence, p and q are constants, and $f(x, y)$ is a linear function. [This proof of Bernstein's theorem is that of J. C. C. Nitsche, *Annals of Mathematics*, 66 (1957), pp. 543–44.]

Minimal surfaces have an extensive literature. See the following expository article:

1. J. C. C. Nitsche, "On new results in the theory of minimal surfaces," *Bulletin of the American Mathematical Society*, 71 (1965), pp. 195–270.